Journal of Statistical Physics, Vol. 50, Nos. 1/2, 1988

Interfacial Properties of a Driven Diffusive System

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Received June 19, 1987; revision received October 21, 1987

A study is made of the low-temperature interfacial properties of a driven system with a single conserved density whose bulk properties were first analyzed, using computer simulations, by Katz, Lebowitz, and Spohn in 1983. The system corresponds to a nearest neighbor interacting lattice gas of charged particles (hence *conserved* order parameter), which are acted upon by a uniform, constant external electric field E. Starting from a bulk kinetic equation, an integral equation for the interface is derived. Nonlocal coupling between different parts of the interface arises from local particle conservation. The interface at any angle is shown to be stable against small deformations of *all* wavelengths that are large compared to the interfacial width. However, the relaxation rate $\omega(\mathbf{k})$ for the interface exhibits a strong *orientational* dependence, which can be understood in terms of the modification of nonlocality by E. The wandering of the interface is considered. Also, the possible stabilizing effect of periodic boundary conditions on the orientation toward the direction of E is discussed.

KEY WORDS: Driven diffusive systems; interface; relaxation rates; nonlocal interactions.

1. INTRODUCTION

There has been considerable interest in the physics of driven diffusive systems in recent years.[1-10] Using the language of critical phenomena, we refer here by the word "diffusive" to a *conserved* order parameter. These systems in their simplest form (for analytic purposes) appear as a nearest neighbor interacting lattice gas of charged particles acted upon by a spatially uniform, temporally constant external electric field.[1] This model is interesting for two reasons: first, it has interesting physical properties, displaying highly anisotropic critical behavior of a new universality class,[6,7] and it has anomalous diffusive behavior for $d \leqslant 2$.[3,4] Second, it

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0022-4715/88/0100-0405$06.00/0

models some real materials that have some technological applications.[11] These materials are commonly known as fast ionic conductors.[12]

The bulk properties of this model have been quite extensively studied, via computer simulations,[1,8] and analytic methods[2,9] on its discrete version, as well as via various analytic techniques[3-7,10] on its continuum version. However, its low-temperature behavior, especially that associated with the interfaces, has yet been investigated. This paper is devoted to such a study.

Although the interfacial properties were not studied in simulations, typical configurations generated by the computer seem to suggest that those configurations with interfaces parallel to the external field E are the most stable ones. This is in contrast to the case of $E=0$, where large clusters would orient in arbitrary directions, reflecting rotational invariance. However, since the sample size is rather small (30), the boundary may stabilize the interface to lie along E, thus masking the genuine effect of E. Moreover, the small size of the sample does not allow a clear-cut separation of length scales.[2] The level of resolution of the simulation results thus far is too low for quantitative measurement of interfacial properties.

In all subsequent sections except Section 5, we consider an interface parallel to E. This paper is organized as follows. In Section 2 the interface equation is derived from the bulk kinetic equation, after making some low-temperature approximations. The properties of the Green's function that mediates nonlocal coupling among different parts of the interface are discussed in Section 3. The relaxation mode $\omega(\mathbf{k})$ under small disturbances about a planar interface is given in Section 4. Section 5 contains a derivation of an equation for a tilted planar interface; and it displays the orientational dependence of the relaxation after deformation. To understand the possible effects of the boundary on the orientation of the interface, we compute in Section 6 the time evolutions of the interface for prescribed initial configurations. The wandering of the interface is computed in Section 7. We conclude this work in Section 8. The Appendix presents a derivation of the Green's function for an interface parallel to E.

2. EQUATION OF MOTION FOR A PLANAR INTERFACE PARALLEL TO E

As in other studies,[3-7] our starting point is again the time-dependent Ginzburg–Landau-type equation of motion, obtained after some coarse

[2] E.g., the lattice constant, the width of the interface, wavelength of interfacial deformation, and a new length introduced by E, which will be defined in Section 3.

graining of the Ising spins over some length scale greater than microscopic scales, but much smaller than the bulk correlation length ξ:

$$\frac{\partial}{\partial t}\phi = -\nabla\cdot[\mathbf{E}\sigma(\phi)] + \lambda[r_{\|}\partial_{\|}^2 + r_{\perp}\nabla_{\perp}^2 - (\nabla^2)^2_{\text{aniso}}]\phi + \lambda\frac{g}{6}\nabla^2\phi^3 + \zeta \tag{2.1}$$

where λ is the kinetic coefficient, E is the external field, $x_{\|}$ is the coordinate along E, $\sigma(\phi)$ is the conductivity, and ζ is the usual Gaussian noise. $(\nabla^2)^2_{\text{aniso}}$ is a short-hand notation for a combination of anisotropic fourth derivatives. In accord with the prevalence of interfaces parallel to E from simulation results, we consider in this and the following sections except Section 5 the interfacial properties of such a parallel interface. We shall assume Ising symmetry for the system throughout this work, so that $\sigma(\phi) = \sigma(-\phi)$.[3] Our primary concern here is its low-temperature properties. Thus, we expand the density variable about a stationary solution corresponding to a planar interface parallel to E:

$$\phi(\mathbf{x}, t) = \phi_c(z) + \psi(\mathbf{x}, t) \tag{2.2}$$

where the "classical" stationary solution is of the form of a kink. As an example, $\phi_c(z) = \phi_\infty \tanh(\xi_0^{-1} z)$, representing a planar interface of width ξ_0.[4] The detailed form of the profile is nevertheless not important in determining the interfacial properties, as the bulk degrees of freedom are averaged over to obtain an interfacial description. As a matter of notation, the d-dimensional coordinates $\mathbf{x}$ are henceforth denoted as

$$\mathbf{x} \equiv (x_{\|}, \mathbf{x}_{\perp}) \equiv (y_{\|}, \mathbf{y}_{\perp}, z)$$

with $\mathbf{x}_{\perp} \equiv (\mathbf{y}_{\perp}, z)$ being the $(d-1)$-dimensional coordinates orthogonal to E, and z being one of these $d-1$ coordinates, arbitrarily chosen, to which the interface is normal. Substituting (2.1) into the kinetic equation results in an equation of motion for the small deviation ψ:

$$\frac{\partial}{\partial t}\psi = \lambda E\phi_c(z)\,\partial_{\|}\psi + \lambda[r_{\|}\partial_{\|}^2 + r_{\perp}\nabla_{\perp}^2 - (\nabla^2)^2_{\text{aniso}}]\psi$$

$$+ \lambda\frac{g}{2}(\partial_{\|}^2 + \nabla_{\perp}^2)\,\phi_c^2(z)\psi + \zeta \tag{2.3}$$

[3] After the submission of this work, I was shown a preprint by A. Hernández-Machado and David Jasnow (Pittsburgh preprint), who worked out the linear stability of the interface parallel to E in the absence of Ising symmetry and the presence of an asymptotic concentration gradient. Their results agree with ours (Section 4) in the appropriate limits.

[4] K.-t. Leung, unpublished results. The author succeeded in fitting a tanh profile of the density for the fully anisotropic bulk equation, provided that certain conditions among the parameters are satisfied. However, he believes that these solutions are not unique.

where we have redefined E such that the term $-E\,\partial_\phi\sigma(\phi_c)$ in (2.1) becomes $E\phi_c$. Now E has the same dimension as the corresponding parameter in the bulk study.[6] The $-\partial_\phi\sigma(\phi_c)$ and ϕ_c differ only in details near the interface. Since we are concerned with phenomena of length scale $\gg\xi$, we make a low-temperature approximation $\phi_c(z)^2\approx\phi_\infty^2$ to get

$$\frac{\partial}{\partial t}\psi\approx\lambda E\phi_c(z)\,\partial_{\|}\psi+\lambda\left[\left(r_{\|}+\frac{g}{2}\phi_\infty^2\right)\partial_{\|}^2+\left(r_\perp+\frac{g}{2}\phi_\infty^2\right)\nabla_\perp^2\right]\psi+\zeta \tag{2.4}$$

where the higher derivatives $(\nabla^2)^2_{\text{aniso}}\psi$ have been dropped because they do not lead to long-wavelength instability, since the coefficients $r_{\|}+\frac{1}{2}g\phi_\infty^2$ and $r_\perp+\frac{1}{2}g\phi_\infty^2$ are generally positive, for $\phi_\infty^2\approx-6r/g$. These coefficients play the role of anisotropic diffusion coefficients. This anisotropy can be removed trivially by rescaling $x_{\|}$ and $\mathbf{x}_\perp$.[5] Calling the rescaled coefficient D, we obtain a diffusion equation with an additional driving force term:

$$\frac{\partial}{\partial t}\psi=\lambda E\phi_c(z)\,\partial_{\|}\psi+D\,\nabla^2\psi+\zeta \tag{2.5}$$

which is supposed to be valid at low temperature for small deviations ψ away from the stationary state ϕ_c.

The interface equation is derived by means of the Green's function technique, following the same line of argument as in Langer and Turski.[13] To do this, we first define the Green's function conjugate to (2.5) by

$$[-\partial/\partial t'-D\nabla'^2+\lambda E\phi_c(z')\partial'_{\|}]\,G(p\,|\,p')=\delta(p-p') \tag{2.6}$$

where for brevity we denote $p=(\mathbf{x},t)=(y_{\|},\mathbf{y}_\perp,z,t)$. The Green's function G vanishes when its arguments go to infinity; and it satisfies causality: $G\propto\theta(t-t')$. By means of Green's theorem,

$$\begin{aligned} D\int_V dV'\,[\psi(p')\nabla'^2G(p\,|\,p')-G(p\,|\,p')\nabla'^2\psi(p')]\\ =D\int_V dV'\,\nabla'\cdot[\psi(p')\nabla'G(p\,|\,p')-G(p\,|\,p')\nabla'\psi(p')]\\ =D\int_{S_+ + S_-} d\mathbf{S}'\cdot[\psi(p')\nabla'G(p\,|\,p')-G(p\,|\,p')\nabla'\psi(p')] \end{aligned} \tag{2.7}$$

where V is the volume of the system, excluding the interface, and S_+ and S_- are the surfaces immediately next to the interface (see Fig. 1). The

[5] However, the anisotropy in the noise correlation cannot be removed simultaneously.

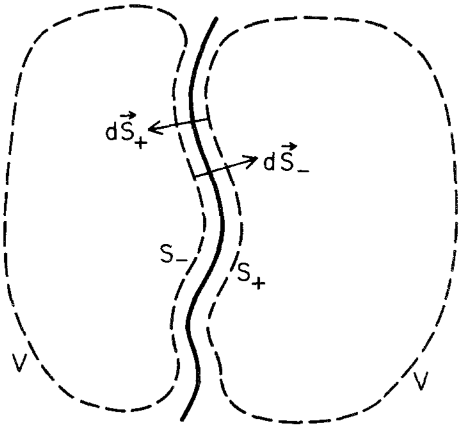

Fig. 1. The bulk volume V, the interface, and the surface S_+ and S_- used in Green's theorem.

contribution from the surface at infinity vanishes because G and ψ vanish there. Since $d\mathbf{S}'$ points in opposite directions for S_+ and S_-, the continuity of $\psi(p')$ across the interface implies that the terms in $\nabla' G$ cancel out. Eliminating $\nabla'^2 G$ and $\nabla'^2\psi$ using the equations of motion for G and ψ and integrating over $\int_{-\infty}^{+\infty} dt'$ gives

$$\begin{aligned}
-D\int_{-\infty}^{+\infty} dt' \int_{S_+ + S_-} d\mathbf{S}' \cdot [G(p \mid p')\,\nabla'\psi(p')] \\
= \int_{-\infty}^{+\infty} dt' \int_V dV' \{\psi(p')[-\partial_{t'} + \lambda E\phi_c(z')\,\partial'_{\|}]\,G(p \mid p') - \psi(p')\,\delta(p-p') \\
- G(p \mid p')[\partial_{t'} - \lambda E\phi_c(z')\partial'_{\|}]\psi + G(p \mid p')\,\zeta(p')\} \\
= \int_{-\infty}^{+\infty} dt' \int_V dV' \,[-\partial t'(\psi G) + \lambda E\phi_c(z')\,\partial'_{\|}(\psi G) + G\zeta] - \psi(p) \\
= \lambda E(\phi_\infty - \phi_{-\infty}) \int_{-\infty}^{+\infty} dt' \int_{S_+} dS' \frac{\partial'_{\|} f'}{\sqrt{g'}}\,\psi G \\
+ \int_{-\infty}^{+\infty} dt' \int dV'\, G\zeta - \psi(p) \qquad (2.8)
\end{aligned}$$

where $\sqrt{g} \equiv [1 + (\nabla f)^2]^{1/2}$ is just the inverse of the local direction cosine on the interface. The lhs can be expressed in terms of the currents across the interface, which are driven by $\nabla\psi$:

$$\begin{aligned}
\text{lhs} &= \int dt' \int_{S_+} dS'\, \hat{\mathbf{n}}' \cdot [\mathbf{J}_{\text{diffusion}}(S_-) - \mathbf{J}_{\text{diffusion}}(S_+)]\, G(p \mid p') \\
&= \int dt' \int_{S_+} dS'\, \hat{\mathbf{n}}' \cdot [\mathbf{J}(S_-) - \mathbf{J}(S_+)]\, G(p \mid p') \\
&\quad - \int dt' \int_{S_+} dS'\, \hat{\mathbf{n}}' \cdot [\mathbf{J}_{\text{induced}}(S_-) - \mathbf{J}_{\text{induced}}(S_+)]\, G(p \mid p') \\
&= -2\phi_\infty \int dt' \int_{S_+} dS' \frac{\dot{f}'}{\sqrt{g'}} G(p \mid p') \\
&\quad + 2\phi_\infty \lambda E \int dt' \int_{S_+} dS' \frac{\partial'_{\parallel} f'}{\sqrt{g'}} \psi(p')\, G(p \mid p') \qquad (2.9)
\end{aligned}$$

where $\hat{\mathbf{n}}'$ is the unit normal to the interface pointing in the $+\hat{z}$ direction. The net flux of particles across the interface in the first term has been related to the motion of the interface. Here $\hat{e}$ is the unit vector along E. Therefore the induced-current contributions to both sides of the equation cancel out.[6]

We now write $dS/\sqrt{g} = d^{d-1}y$, and use the Gibbs–Thomson relation to express the deviation $\psi(S_+)$ at the interface as $2\phi_\infty \xi \mathscr{K}$. Here

$$\mathscr{K} = -\frac{\delta}{\delta f(\mathbf{y}, t)} \int dS' = -\frac{\delta}{\delta f(\mathbf{y}, t)} \int d^{d-1}y' \sqrt{g'}$$

is the curvature of the interface at $f(\mathbf{y}, t)$. We eventually obtain the equation of motion for the interface in the form of an integral equation:

$$\begin{aligned}
&-\xi \frac{\delta}{\delta f(\mathbf{y}, t)} \int d^{d-1}y' \sqrt{g'} \\
&\quad = \int_{-\infty}^{+\infty} dt' \int d^{d-1}y'\, G(\mathbf{y}, t \mid \mathbf{y}', t'; f \mid f') \frac{\partial}{\partial t'} f(\mathbf{y}', t') \\
&\quad\quad + \frac{1}{2\phi_\infty} \int_{-\infty}^{+\infty} dt' \int dV'\, G(\mathbf{y}, t \mid \mathbf{y}', t'; f \mid z')\, \zeta(\mathbf{y}', z', t') \qquad (2.10)
\end{aligned}$$

Most of the subsequent analysis is based on this equation. It is in the same form as that of Ref. 13, with a different Green's function, which contains all the E dependence. Also, a noise term is added,[14] which will be necessary in calculating any correlation function.

[6] This cancellation is generally true for $\mathbf{J}_{\text{induced}}(\phi)$ even in ϕ, which holds for systems having particle–hole symmetry. The case when this current is odd in ϕ has not yet been explored.

Before we close this section, a few comments are in order here: (1) This equation is highly nonlinear: nonlinearities are contained both in the curvature term $(\delta/\delta f) \int d^{d-1}y \sqrt{g}$ and in G. (2) The nonlocal nature of the interaction among the interfacial degrees of freedom mediated by the Green's function is a general feature for systems with local conservation. Since G decays exponentially, the interaction is short-ranged. (3) Given the correlation of the bulk noise ζ, it is straightforward within the linear approximation to work out that of the noise on the interface. This is given in Section 7.

3. GREEN'S FUNCTIONS

In this section we briefly discuss the functional form of the Green's functions for an interface parallel to E. Note first that the interfacial degrees of freedom interact in a nonlocal way, the extent of which is specified by G. Now, G contains *all* the E dependence for the interface because the noise correlation is also related to G. Thus, it is essential to know its explicit functional form for an understanding of the influence of both the local conservation and E on the interaction of the interfacial degrees of freedom. Since we will see that the planar interface is stable against small deformations of all wavelengths, it is sufficient to consider only the linearized equation of motion. So $G(\mathbf{y}, t \mid \mathbf{y}', t'; z=0 \mid z'=0)$ should capture the essential qualitative features of the nonlocal coupling. For comparison, we also show the corresponding expressions below for the undriven case of model A (nonconserved order parameter) and model B (conserved order parameter), following the nomenclature of Hohenberg and Halperin.[19] In momentum-frequency space, one finds (see the Appendix)

$$G(\mathbf{k},\ \omega; 0 \mid 0) = \begin{cases} \text{const} & \text{(model A)} \\ [2D(k^2 - iD^{-1}\omega)^{1/2}]^{-1} & \text{(model B)} \\ D^{-1}[(k^2 - iD^{-1}\omega - ik_E k_{\|})^{1/2} & \\ \qquad + (k^2 - iD^{-1}\omega + ik_E k_{\|})^{1/2}]^{-1} & (E \neq 0) \end{cases} \tag{3.1}$$

where $\mathbf{k}$ is a $(d-1)$-dimensional wavevector, and $k_E \equiv \lambda D^{-1} E \phi_\infty$ is a wavenumber introduced by the driving force E. The square roots are defined as $(Re^{i\theta})^{1/2} = R^{1/2} e^{i\theta/2}$, where θ is as shown in Fig. 2. Fourier-transforming to real time, we have $G(\mathbf{k}, t \mid t'; 0 \mid 0) = 0$ for $t - t' < 0$, due to the fact that G has no singularity in the upper half ω-plane, which reflects the causality condition satisfied by G. For $t - t' > 0$, we get

$$G(\mathbf{k}, t \,|\, t'; 0 \,|\, 0) = \begin{cases} \text{const} \times \delta(t-t') & \text{(model A)} \\ \left(\dfrac{1}{4\pi D(t-t')}\right)^{1/2} \exp[-k^2 D(t-t')] & \text{(model B)} \\ \left(\dfrac{1}{4\pi D(t-t')}\right)^{1/2} \exp[-k^2 D(t-t')] & \\ \quad \times \dfrac{\sin[k_E k_{\|} D(t-t')]}{k_E k_{\|} D(t-t')} & (E \neq 0) \end{cases} \tag{3.2}$$

where we have computed exactly the $\int d\omega$ for the $E \neq 0$ case by distorting the contour of integration into the lower half ω-plane and then integrating the contributions along the four branch cuts. Further Fourier-transforming to real space $\mathbf{y} = (y_{\|}, \mathbf{y}_{\perp})$ yields

$$G(\mathbf{y}, t \,|\, \mathbf{y}', t'; 0 \,|\, 0) = \begin{cases} \text{const} \times \delta^{d-1}(\mathbf{y}-\mathbf{y}')\, \delta(t-t') & \text{(model A)} \\ \left(\dfrac{1}{4\pi D(t-t')}\right)^{d/2} \exp \dfrac{-|\mathbf{y}-\mathbf{y}'|^2}{4D(t-t')} & \text{(model B)} \\ \left(\dfrac{1}{4\pi D(t-t')}\right)^{d/2} \left(\exp \dfrac{-|\mathbf{y}-\mathbf{y}'|^2}{4D(t-t')}\right) & \\ \quad \times J(y_{\|} - y'_{\|}, t-t') & (E \neq 0) \end{cases} \tag{3.3}$$

where

$$J(y_{\|}, t) \equiv \frac{1}{2} \int_{-1}^{+1} d\alpha \exp(-y_{\|} k_E \alpha/2 - k_E^2 D t \alpha^2/4) \tag{3.4}$$

Hence all E dependence is contained in a single multiplicative factor J, which represents the effect of the driving force on the nonlocal interaction

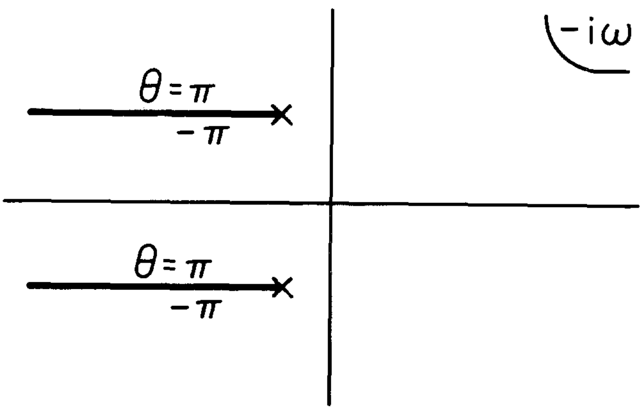

Fig. 2. The branches and branch cuts for G for $E \neq 0$. The branch cuts merge into the real axis as $E \to 0$.

along the $y_{||}$ direction. Associated with the length scale k_E^{-1}, E introduces a new time scale $\tau_E \equiv (Dk_E^2)^{-1}$. It is then natural to consider the limiting forms of J for $t-t' \ll \tau_E$ and $t-t' \gg \tau_E$:

1. $t-t' \ll \tau_E$. For $y_{||}=0$

$$J(0, t) = \frac{1}{2}\int_{-1}^{+1} d\alpha \exp(-k_E^2 D t \alpha^2/4) \approx 1$$

Thus, $G \propto \exp[-y_{||}^2/4D(t-t')]$ as in model B.

For $y_{||} \neq 0$

$$\begin{aligned} J(y_{||}, t) &\approx \frac{1}{2}\int_{-1}^{+1} d\alpha \exp(-y_{||} k_E \alpha/2) \\ &= (\sinh A)/A \\ &= \begin{cases} 1 + A^2/6 + \cdots, & A \ll 1 \\ \frac{1}{2}|A|^{-1} e^{|A|} + \cdots, & |A| \gg 1 \end{cases} \end{aligned} \tag{3.5}$$

where we denote $y_{||}k_E/2$ as A. So, for $|y_{||}| \gg k_E^{-1}$ the level of nonlocal interaction is enhanced by a factor $(1/|y_{||}|k_E)\exp(|y_{||}|k_E/2)$. Significant modification to the pure model B behavior of the form $\exp[-y_{||}^2/4D(t-t')]$ occurs only for length $|y_{||}| > k_E^{-1}$. Conversely, suppression of nonlocality is observed for $t-t' > \tau_E$ as follows:

2. $t-t' \gg \tau_E$. We consider three different regions:

(i) For $|y_{||}| \ll k_E^{-1}$, we get

$$J(y_{||}, t) \approx \frac{\pi^{1/2}}{k_E[D(t-t')]^{1/2}} = \left(\frac{\pi\tau_E}{|t-t'|}\right)^{1/2} \ll 1$$

Hence the level of nonlocal interaction is suppressed by this factor.

(ii) For

$$k_E^{-1} \ll |y_{||}| \ll k_E D(t-t') = [D(t-t')]^{1/2}[(t-t')/\tau_E]^{1/2}$$

we get the same suppression from J as in region (i).

(iii) For $k_E D(t-t') \ll |y_{||}|$, we get

$$\begin{aligned} J(y_{||}, t) &\approx \frac{1}{2}\int_{-1}^{+1} d\alpha \exp(-k_E y_{||}\alpha/2) \\ &= (\sinh A)/A \\ &\approx \frac{1}{2}|A|^{-1}\exp|A| \\ &= (1/|y_{||}|k_E)\exp(|y_{||}|k_E/2) \gg 1 \end{aligned} \tag{3.6}$$

Hence, only when $|y_{||}|$ goes beyond the large length scale $k_E D(t-t')$ set by E would the model B behavior be modified to

$$G \propto (1/|y_{||}|k_E) \exp(|y_{||}|k_E/2) \exp[-y_{||}^2/4D(t-t')] \tag{3.7}$$

which represents an enhancement of nonlocality over the model B behavior at the tail for large $|y_{||}|$.

Figure 3 displays two typical examples of a comparison between $G(E)$ and $G(E=0)$ for $t-t' < \tau_E$ and $t-t' > \tau_E$. To summarize, the new length scale k_E^{-1} and time scale τ_E mark the borders beyond which deviations from

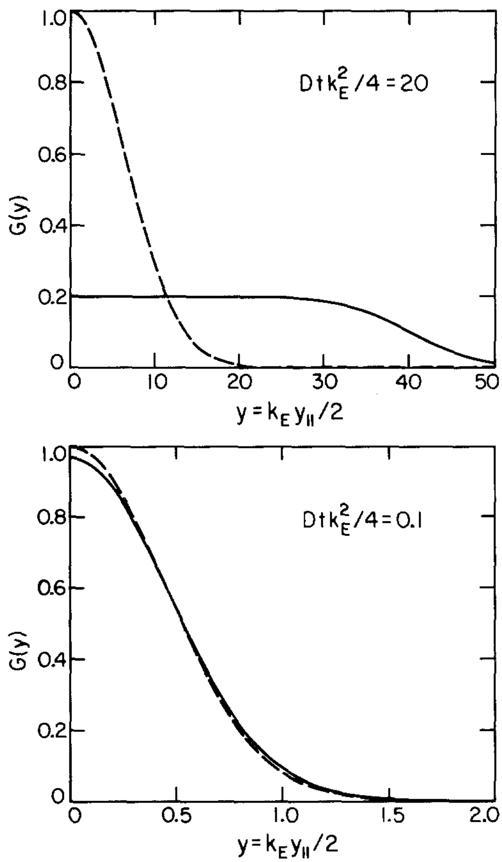

Fig. 3. Two typical examples of the behavior of the Green's function $G(\mathbf{y}, t|\mathbf{0}, 0; 0|0)$ along the direction of E, (a) for $t > \tau_E$ and (b) for $t < \tau_E$, showing the suppression at short distances and enhancement at long distances of the nonlocal coupling due to E (solid line), in comparison to that of $E=0$ (dashed lines). The magnitudes of G's for $E=0$ are properly normalized to 1 at $y_{||}=0$.

model B behavior are observed. These modifications help to explain in an intuitive way the new relaxational behavior to be discussed in the next section.

4. STABILITY OF THE PLANAR INTERFACE

The planar interface $f=0$ is obviously a stationary solution of the interface equation of motion corresponding to $\dot{f}=0$.[7] To examine its stability against *small* deformations, we linearize[8] (2.10) with respect to f and look for a solution of the form

$$f(\mathbf{y}, t)=f_0(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{y}-i\omega t)$$

where $\mathbf{k}$ is $(d-1)$-dimensional. Substituting this into (2.10) yields

$$-\xi k^2 f_0(\mathbf{k})=G(\mathbf{k}, \omega; 0\,|\,0)[-i\omega f_0(\mathbf{k})]$$

By (2.11), the dispersion relation determining $\omega(\mathbf{k})$ is thus given by

$$-\xi k^2=\tilde{\omega}[(k^2+\tilde{\omega}+ik_E k_{||})^{1/2}+(k^2+\tilde{\omega}-ik_E k_{||})^{1/2}]^{-1} \qquad (4.1)$$

where $\tilde{\omega}\equiv -iD^{-1}\omega$, so that $f(\mathbf{k}, t)\propto\exp(D\tilde{\omega}t)$. Without loss of generality, we assume henceforth that $E>0$, $\phi_\infty>0$, so $k_E\equiv\lambda D^{-1}E\phi_\infty$ can be taken as positive. Let us study this equation in the complex $\tilde{\omega}$ plane. There are two branch points at $b\equiv -k^2+ik_E k_{||}$ and $b^*= -k^2-ik_E k_{||}$ besides the one at infinity. The branch cuts are chosen to extend from the branch points to $-\infty$ parallel to the real axis (as shown in Fig. 2). A close examination shows that (4.1) can be satisfied at one and only one point on the *negative* real axis of $\tilde{\omega}$. Therefore, the planar interface parallel to E is stable for all wavevectors $\mathbf{k}$. The relaxation mode $\tilde{\omega}(\mathbf{k})$ is given by one of the roots of the following equation, which is obtained by squaring (4.1) twice:

$$\tilde{\omega}^4-4k^4\xi^2\tilde{\omega}^3-4k^6\xi^2\tilde{\omega}^2-4(k^4\xi^2 k_E k_{||})^2=0 \qquad (4.2)$$

This has two real roots and two complex roots; only the real and negative root is physical. We are primarily interested in the relaxation modes under a long-wavelength deformation, so the small-k behavior of (4.2) is considered. The finding of an analytic expression for the roots of (4.2) can be simplified by the following observation: simple power counting shows that

[7] There are no cellular states as in Ref. 15, due to the absence of a concentration gradient at $\pm\infty$.

[8] From the explicit exponentially decaying form of G, such a linearization overestimates the strength of coupling for large deformations f.

the odd term proportional to $\tilde{\omega}^3$ is always the least dominant one. It could only introduce correction in powers of k^2. We thus drop it, and the algebraic equation becomes quadratic in $\tilde{\omega}^2$, which can be easily solved to give

$$\begin{aligned}\tilde{\omega}^2 &\approx 2k^6\xi^2 + [(2k^6\xi^2)^2 + (2\xi^2 k_E k_{\|} k^4)^2]^{1/2} \\ &= 2k^6\xi^2 + 2k^4\xi^2[k^4 + (k_E k_{\|})^2]^{1/2}\end{aligned} \tag{4.3}$$

of which the negative root is the relaxation mode for a small deformation of wave vector $\mathbf{k}$. Note that the new length scale k_E^{-1} always couples to $k_{\|}$, and that the ratio of the two terms inside the square roots varies over a large range of values as E changes:

$$\begin{aligned}\text{ratio}^{1/2} &= \frac{k_E|k_{\|}|}{k^2} = \frac{k_E|k_{\|}|}{k_{\|}^2 + k_{\perp}^2} \\ &\approx \begin{cases}(k_E/|k_{\perp}|)(|k_{\|}|/|k_{\perp}|) & \text{for} \quad k_{\|}^2 \ll k_{\perp}^2 \\ (k_E/|k_{\|}|) & \text{for} \quad k_{\|}^2 \gg k_{\perp}^2\end{cases}\end{aligned} \tag{4.4}$$

we consider two limiting cases:

1. Small E, such that $k_E \ll |k_{\|}|$ and $k_E \ll |k_{\perp}|$. Here the relaxation occurs on scales so much smaller than k_E^{-1} that the presence of E is hardly felt by the interface. The relaxation modes for $(|k_{\|}|/|k_{\perp}|)(k_E/|k_{\perp}|) \ll 1$ (i.e., $|k_{\|}|/|k_{\perp}|$ is not too large) take the usual form of model B:

$$\omega(\mathbf{k}) = -2Dk^3\xi - \tfrac{1}{4}Dk_E^2 k_{\|}^2 k^{-1}\xi + O(k_E^4) \tag{4.5}$$

where we denote the relaxation rate as $\omega(\mathbf{k})$: $f(\mathbf{k}, t) \propto \exp[\omega(\mathbf{k})t]$. The leading term of (4.5) is just the result of Langer and Turski[13] in the absence of asymptotic concentration gradient. The term proportional to E^2 is of order $k_{\|}$ in the physically interesting case of $k_{\|}^2 \gg k_{\perp}^2$, whose negative sign simplies a faster decay than model B. Obviously, since k_E always couples to $k_{\|}$, the same k^3 behavior is observed whenever the deformation varies only in the orthogonal directions, i.e., when $k_{\|} = 0$. So from now on we only consider $k_{\|} \neq 0$.

2. Large E, such that $k_E \gg |k_{\|}|$ and $k_E \gg |k_{\perp}|$. The general expression simplifies when $k_E|k_{\|}|/k^2 \gg 1$, which is true when $|k_{\|}|/|k_{\perp}|$ is not too small:

$$\omega(\mathbf{k}) \approx -(2k_E)^{1/2} D\xi k^2 |k_{\|}|^{1/2} - \frac{Dk^4\xi}{(2k_E k_{\|})^{1/2}} \tag{4.6}$$

When the deformation varies predominantly along E, we have $k_{\|}^2 \gg k_{\perp}^2$:

$$\omega(\mathbf{k}) \approx -(2k_E)^{1/2} D\xi k_{\|}^{5/2} \tag{4.7}$$

which shows that the presence of large E is approximately realized as $k_E^{1/2}$ replacing $k_{\|}^{1/2}$ in the relaxation mode, resulting in a faster relaxation.

Within the linear approximation we can superpose two small deformations: one varies predominantly along E and the other orthogonal to E, with characteristic wavelength denoted as l. The above results indicate that at later times, of the order of $t \gg [(2k_E)^{1/2}\xi l^{-5/2}]^{-1}$, the deformation along E would decay to an amplitude negligible in comparison to that orthogonal to E. Similar faster temporal decay of fluctuations along E was also found near the critical point as in Ref. 6.

To conclude this section, let us compare the relaxation for model A (i.e., without local conservation), for model B (i.e., $E=0$ with local conservation), and for the case when $E \neq 0$, expanded in small k:

$$\omega(\mathbf{k}) \approx \begin{cases} -\Gamma k^2 & \text{(model A)} \\ -2D\xi k^3 & \text{(model B)} \\ -(2k_E)^{1/2} D\xi k^2 |k_{\|}|^{1/2} & (E \neq 0) \end{cases}$$

Mathematically, the origin of the slower decay characterizing a system with conservation is contained in the $(\mathbf{k}, \omega)$ dependence of the Green's function. As an example, $\omega(\mathbf{k})$ for model B is determined by

$$-\xi k^2 = \frac{\omega(\mathbf{k})}{2D[D^{-1}\omega(\mathbf{k}) + k^2]^{1/2}}$$

For small k the k^2 inside the square root dominates, giving $\omega(\mathbf{k}) \propto -k^3$. In model A the entire denominator is absent, hence $\omega(\mathbf{k}) \propto -k^2$.

The Green's function describes nonlocal interaction among interfacial degrees of freedom at different space-time points for systems with locally conserved density. Physically, such nonlocality arises from the necessity of the transport of materials in modifying the interface. Deformations at different space-time points do not relax independently; they are coupled together through G. Therefore, a better understanding can be gained by examining the explicit form of $G(\mathbf{y}_\perp, y_{\|}, t \,|\, \mathbf{y}'_\perp, y'_{\|}, t'; 0\,|\,0)$ as in Section 3, where we have shown that for any given $t-t'$, the coupling in the corresponding important range along E, as determined by the model B behavior $\exp[-|y_{\|}-y'_{\|}|^2/4D(t-t')]$, is suppressed by the factor J, whereas the coupling is enhanced for large $|y_{\|}-y'_{\|}|$. Along $\mathbf{y}^\perp$, the nonlocality decays as in model B, with $G \propto \exp[-|\mathbf{y}_\perp - \mathbf{y}'_\perp|^2/4D(t-t')]$. To see this effect of E on the relaxation, let us consider the linearized equation of motion

$$-\xi \nabla^2 f(\mathbf{y}, t) = \int dt'\, d^{d-1}y\, G(\mathbf{y}, t \,|\, \mathbf{y}', t'; 0\,|\,0)\, \dot{f}(\mathbf{y}', t') \tag{4.8}$$

In the momentum space, a deformation with a definite wavevector is just sinusoidal in shape. For model A,

$$G(\mathbf{y}, t \mid \mathbf{y}', t'; 0 \mid 0) \propto \delta^{d-1}(\mathbf{y}-\mathbf{y}')\,\delta(t-t')$$

the interaction is local, so that the velocity $\dot{f}(\mathbf{y}, t)$ directly responds to the curvature of f at $(\mathbf{y}, t)$. This leads to $\omega(\mathbf{k}) \propto -k^2$.

Now if there is small nonlocality, as arising from local particle conservation in model B, the $\dot{f}$ at neighboring points $(\mathbf{y}', t')$ of $(\mathbf{y}, t)$, which have the same sign as $\dot{f}(\mathbf{y}, t)$, would also contribute to the integral on the rhs of (4.8) and it competes with $\dot{f}(\mathbf{y}, t)$. This decreases $\dot{f}(\mathbf{y}, t)$ relative to that of model A. The specific $(\mathbf{k}, \omega)$ dependence of G then gives $\omega(\mathbf{k}) \propto -k^3$ in the small-k limit.

For large E, consider a particular Fourier mode of deformation (sinusoidal) of a definite wavevector $\mathbf{k} = (\mathbf{k}_\perp = \mathbf{0}, k_\parallel)$ with $k_\parallel \ll k_E$, since anomalous decay is observed only when $k_\parallel^2 \gg k_\perp^2$ and $k_\parallel \ll k_E$. The situation is illustrated in Fig. 4. Modification of model B relaxation is a direct consequence of the suppression of nonlocality at short distances (e.g., within the crest "a"), and enhancement at long distances (the dominant regions of which are those in the neighboring valleys "b" and "c"), where the interface velocities have opposite sign to that at "a." Both these two effects thus contribute to speed up the relaxation of the interface relative to that of model B. The detailed $(\mathbf{k}, \omega)$ dependence of G then determines $\omega(\mathbf{k})$ as in (4.3). This heuristic argument illustrates that any nontrivial modification of model B behavior by E can only be observed at long time and large longitudinal distance compared to those scales introduced by E.

Intuitively one does not expect E to change the relaxation along the orthogonal directions, because the induced current is unidirectional along

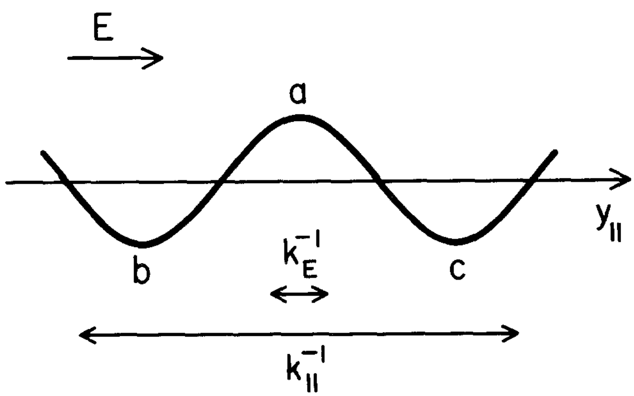

Fig. 4. A sinusoidal deformation of wavelength $k_\parallel^{-1}$ of the plane interface. In the case of large E, $k_\parallel \ll k_E$.

E, reflecting in the coupling to $k_{\|}$ alone in the equation of motion. This transport of particles along E somehow competes with the usual model B nonlocal interaction. This manifests explicitly in the competition between the two terms inside the square root of (4.3).

5. STABILITY OF A TILTED INTERFACE

This is the only section that deals with an interface *not* parallel to E. In Section 2 the interface equation of motion is derived by expanding the bulk equation about a stationary state, which describes a planar interface parallel to the driving force. The planar interface is subsequently shown to be stable against small deformation. On the other hand, it was found[19] that there exist stationary solutions that correspond to a tilted interface. Nevertheless, its stability is not yet explored, due to the complexity inherent in analyzing a fourth-order nonlinear differential equation. We therefore derive in this section a kinetic equation for the interface by expanding the bulk equation about a presumed solution $\phi_c(u)$, which describes a tilted interface. Let

$$\phi(\mathbf{x}, t) = \phi_c(u) + \psi(\mathbf{x}, t)$$

where ψ represents the deviation about $\phi_c(u)$, with $u = (z - f_1 y_{\|})/\sqrt{g_1}$ and $g_1 = 1 + f_1^2$. As before, f_1 is the tangent of the angle of inclination. For small deviation ψ the bulk equation can be linearized to get

$$\frac{\partial}{\partial t}\psi = \lambda E \phi_c(u)\,\partial_{\|}\psi - \lambda E \frac{f_1}{\sqrt{g_1}}\phi_c'(u)\psi + (D_{\|}\partial_{\|}^2 + D_{\perp}\nabla_{\perp}^2)\psi + \zeta \quad (5.1)$$

in which fourth derivatives are dropped because the effective diffusion coefficients $D_{\|} \equiv r_{\|} + \frac{1}{2} g\sigma_{\|}\phi_{\infty}^2$ and $D_{\perp} \equiv r_{\perp} + \frac{1}{2} g\sigma_{\perp}\phi_{\infty}^2$ are both positive, where $\sigma_{\perp}$ and $\sigma_{\|}$ account for anisotropy in the derivatives coupled to $g\phi^3$. For simplicity, from now on we only consider the situation in two dimensions; extension to higher dimensions should be straightforward. It is natural to perform a coordinate transformation from $(y_{\|}, z)$ to (u, v), where u measures the normal distance from the planar interface, and $v \equiv g_1^{-1/2}(y_{\|} + f_1 z)$ measures along the interface, as shown in Fig. 5. This is the intrinsic coordinates for the interface. Denote $\psi(y_{\|}(u, v), z(u, v))$ as $\varphi(u, v)$; (5.1) becomes

$$\frac{\partial}{\partial t}\varphi = -\lambda E \frac{f_1}{\sqrt{g_1}}\partial_u(\phi_c(u)\varphi) + \frac{\lambda E}{\sqrt{g_1}}\phi_c(u)\,\partial_v\varphi + \left(D_u\partial_u^2 - 2\Delta D\frac{f_1}{g_1}\partial_u\partial_v + D_v\partial_v^2\right)\varphi + \zeta \quad (5.2)$$

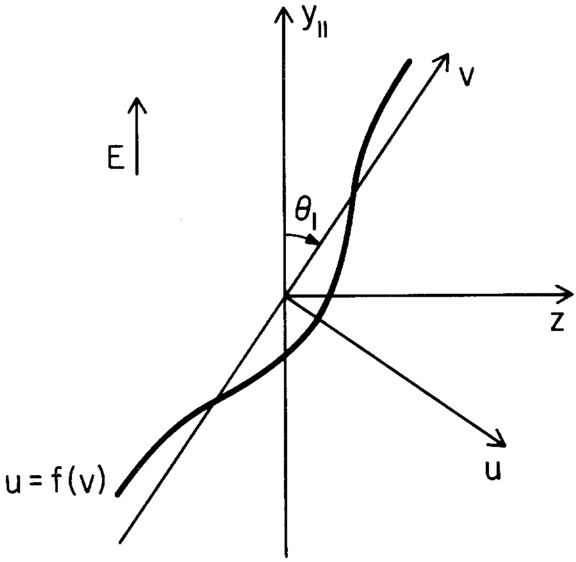

Fig. 5. The tilted planar interface, its deformation, and its intrinsic coordinates. Particles are chosen to reside on the side of positive u.

where $\Delta D \equiv D_{\|} - D_{\perp}$, $D_u \equiv D_{\|} - \Delta D/g_1 > 0$, and $D_v \equiv D_{\|} - f_1^2 \Delta D/g_1 > 0$. We first diagonalize (5.2) with respect to v by Fourier transform; thus,

$$\frac{\partial}{\partial t} \varphi(u, k_v, t) = -\lambda E \frac{f_1}{\sqrt{g_1}} \partial_u(\phi_c(u)\varphi) + \lambda E \frac{1}{\sqrt{g_1}} ik_v \phi_c(u)\varphi + \left(-D_v k_v^2 + 2\Delta D \frac{f_1}{g_1} ik_v \partial_u + D_u \partial_u^2\right)\varphi + \zeta \tag{5.3}$$

whose corresponding Green's function is acted on by its conjugate operator:

$$\left[-\partial_t - D_u \partial_u^2 + 2\Delta D \frac{f_1}{g_1} \partial_u \partial_v - D_v \partial_v^2 + \lambda E \frac{1}{\sqrt{g_1}} \phi_c(u) \partial_v - \lambda E \frac{f_1}{\sqrt{g_1}} \phi_c(u) \partial_u\right] G(p' \mid p) = \delta(p' - p) \tag{5.4}$$

Here $p \equiv (u, v, t)$. Exactly the same procedure as in Section 2 leads to an integral equation for the interface position $f(v, t)$. Again, the term proportional to E cancel out, leaving

$$-\xi \frac{\delta}{\delta f(v, t)} \int dv' \, [1 + (\partial_{v'} f')^2]^{1/2}$$
$$= \int_S dt' \, dv' \, G(v, f, t \,|\, v' f', t') \, \dot{f}(v', t')$$
$$+ \frac{1}{2\phi_\infty} \int_V dt' \, du' \, dv' \, G(v, f, t \,|\, v', u', t') \, \zeta(v', u', t') \tag{5.5}$$

Now we first have to solve (5.4) for G, and then study the stability of the $f = 0$ solution of the linearized equation. We shall not go through the details finding G, which is nothing more than matching exponential solutions at boundaries. Again a low-temperature approximation $\phi_c(u) \approx \phi_\infty[2\theta(u) - 1]$ is used. To linear order in f, we only need

$$G(k_v, \omega; u = 0 \,|\, u' = 0) = \frac{1}{D_u} \frac{1}{q + p} \tag{5.6}$$

where p and q are the roots of the following quadratic equations with positive real parts:

$$D_u p^2 - \left(2\Delta D \frac{f_1}{g_1} ik_v + D_{\|} k_E \frac{f_1}{\sqrt{g_1}} \right) p - \left(D_v k_v^2 - i\omega - \frac{1}{\sqrt{g_1}} D_{\|} ik_E k_v \right) = 0 \tag{5.7}$$

$$D_u q^2 + \left(2\Delta D \frac{f_1}{g_1} ik_v - D_{\|} k_E \frac{f_1}{\sqrt{g_1}} \right) q - \left(D_v k_v^2 - i\omega + \frac{1}{\sqrt{g_1}} D_{\|} ik_E k_v \right) = 0 \tag{5.8}$$

It is then easy to get

$$G(k_v, \omega; 0 \,|\, 0) = \left\{ \frac{D_{\|} k_E f_1}{\sqrt{g_1}} + \left[D_{\|} D_\perp k_v^2 + \left(\frac{D_{\|} k_E f_1}{2\sqrt{g_1}} \right)^2 - \frac{1}{\sqrt{g_1}} D_{\|} D_\perp ik_E k_v - D_u i\omega \right]^{1/2} + \left[D_{\|} D_\perp k_v^2 + \left(\frac{D_{\|} k_E f_1}{2\sqrt{g_1}} \right)^2 + \frac{1}{\sqrt{g_1}} D_{\|} D_\perp ik_E k_v - D_u i\omega \right]^{1/2} \right\}^{-1} \tag{5.9}$$

and the relaxation rate $\omega(k_v)$ in $\exp[\omega(k_v) t]$ is obtained by solving

$$-\xi k_v^2 = \omega(k_v) \, G(k_v, i\omega(k_v); 0 \,|\, 0) \tag{5.10}$$

To see whether (5.10) has a solution for $\omega(k_v)$ with positive real part, we denote $D_u\omega(k_v) = \omega_r + i\omega_i$. The square roots are now of the form

$$R_{\pm}^{1/2} e^{i\theta_{\pm}/2} = [a^2 + b^2 + \omega_r + i(\omega_i \pm c)]^{1/2}$$

where we denote $D_{\|} k_E f_1/(2\sqrt{g_1})$ as b, etc. To satisfy (5.10) with $\omega_r > 0$, we must have Re $G < 0$. However,

$$\begin{aligned}
\operatorname{Re} G^{-1} &= 2b + R^{1/2}\cos(\theta/2) \\
&= 2b + [R_+(1+\cos\theta_+)/2]^{1/2} + [R_-(1+\cos\theta_-)/2]^{1/2} \\
&= 2b + (1/\sqrt{2})\{[(a^2+b^2+\omega_r)^2 + (\omega_i + c)^2]^{1/2} + a^2 + b^2 + \omega_r\}^{1/2} \\
&\quad + (1/\sqrt{2})\{[(a^2+b^2+\omega_r)^2 + (\omega_i - c)^2]^{1/2} + a^2 + b^2 + \omega_r\}^{1/2}
\end{aligned}$$

which shows that, for $\omega_r > 0$, Re $G^{-1} > 2b + 2(a^2 + b^2)^{1/2} > 0$ for *all* values of b. Therefore, there cannot be solution with $\omega_r > 0$. Thus, a tilted interface is always stable against small deformation of any wavelength k_v, at any orientation f_1. We expect that this result also holds in higher dimensions.

Relaxation Modes

Let us now use (5.10) to determine the relaxation modes for large-wavelength deformation. We shall look for a real $\omega(k_v)$ solution.

Let $\Omega = \omega(k_v)/k_v^2\xi$ and $b = D_{\|} k_E f_1/2\sqrt{g_1}$. Dropping higher order terms in k_v and squaring (5.10) twice, we obtain the relaxation mode from the negative root of an algebraic equation

$$\begin{aligned}
&\frac{1}{4}\Omega^4 + b\Omega^3 + (b^2 - D_{\|} D_{\perp} k_v^2)\Omega^2 + 2b(2b^2 - D_{\|} D_{\perp} k_v^2)\Omega \\
&\quad - D_{\|}^2 D_{\perp}^2 \frac{1 + D_{\|} f_1^2/D_{\perp}}{1 + f_1^2} k_E^2 k_v^2 = 0
\end{aligned} \tag{5.11}$$

For infinitesimal inclination f_1 the leading terms are the first and the last one, giving the $k_v^{5/2}$ behavior found in Section 4. For any finite f_1, b^2 is much greater than k_v^2. For finite $f_1 > 0$, so $b > 0$, (5.11) gives

$$\Omega = 4b + O(k_v^2) \tag{5.12}$$

leading to an $\omega(k_v) \propto -k_v^2$ behavior. On the other hand, for finite $f_1 < 0$, we find

$$\Omega \approx \frac{1}{4b^3} D_{\|}^2 D_{\perp}^2 k_E^2 k_v^2 \xi \frac{1 + D_{\|} f_1^2/D_{\perp}}{1 + f_1^2} \tag{5.13}$$

corresponding to $\omega(k_v) \propto -k_v^4$. Since the anisotropy in the diffusion coefficients are not crucial, we simply denote both by D and the results are neatly summarized as

$$\omega(k_v) = -Dk_v^2\, \xi k_E(f_1/\sqrt{g_1})\, F(k_v^2/k_E^2, f_1/\sqrt{g_1}) \tag{5.14}$$

where $F(x, y)$ has the following asymptotic behavior for $x \ll 1$:

$$F(x, y) \approx \begin{cases} 2^{1/2} x^{1/4} y^{-1} & y \to 0 \\ 2 + O(x) & y = O(1) > 0 \\ -2xy^{-4} & y = O(1) < 0 \end{cases} \tag{5.15}$$

Equations (5.14) and (5.15) are the principal results of this section. In particular, this implies that for the configuration of Fig. 6a, the relaxation behaves like model A, whereas for that of Fig. 6b, it is even slower than model B.

Physically we have seen that the relaxation is strongly influenced by the range and strength of the nonlocal interaction, which is specified entirely by the Green's function. We should therefore gain better understanding of the above anomalous behavior by studying the detailed form of $G(\mathbf{y}, t)$. To illustrate, let us evaluate those for the two limiting cases depicted in Fig. 6, when the interface is orthogonal to the driving force. In these configurations G is obviously isotropic with respect to the transverse $(d-1)$ dimensions. From (5.9) we get, after generalizing k_v to $(d-1)$-dimensional $\mathbf{k}$,

$$G(\mathbf{k}, \omega; 0|0; f_1 = \pm\infty) = \left[\pm D_{\parallel} k_E + 2\left(D_{\parallel} D_{\perp} k^2 + \frac{1}{4} k_E^2 D_{\parallel}^2 - iD_{\parallel}\omega \right)^{1/2} \right]^{-1} \tag{5.16}$$

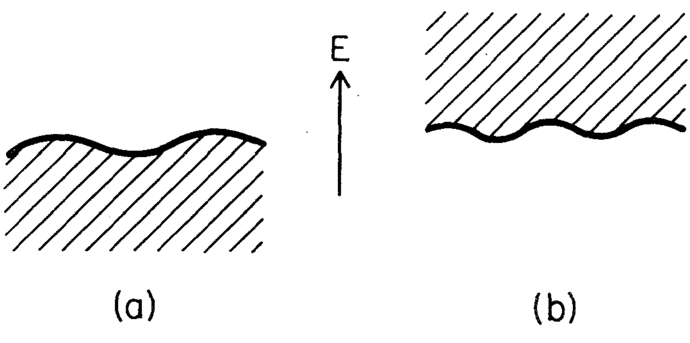

Fig. 6. (a) When $\theta = 90°$, the relaxation is model A like; whereas (b) when $\theta = -90°$, it behaves as k_v^4. The shaded areas correspond to particle-rich phase.

and

$$G(\mathbf{k}, t; 0|0; f_1 = \pm\infty) = \frac{1}{2}\int_C \frac{ds}{2\pi i} e^{sD_{\parallel}t} \frac{1}{\pm k_E/2 + (D_\perp k^2/D_\parallel + \frac{1}{4}k_E^2 + s)^{1/2}} \tag{5.17}$$

There are branch points at $s = -D_\perp k^2/D_\parallel - \frac{1}{4}k_E^2$ and at infinity; and only for $f_1 < 0$ is there a pole at $-D_\perp k^2/D_\parallel$. The appearance of this pole is the mathematical reason for the ultraslow decay $\omega(k_v) \sim -k_v^4$ found for $f_1 < 0$, as we will see briefly. The contour integrals are easily evaluated in a standard way. For $t < 0$, $G = 0$ as usual. The results for $t > 0$ are

$$\begin{aligned} G(\mathbf{k}, t; 0|0) = {} & \tfrac{1}{2}\theta(-f_1) k_E e^{-D_\perp k^2 t} \\ & - (4\pi D_\parallel t)^{-1/2} \exp(-k^2 D_\perp t - k_E^2 D_\parallel t/4)\, K(k_E^2 D_\parallel t/4) \end{aligned} \tag{5.18}$$

where $\theta(x)$ is the step function, and

$$K(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty dy\, e^{-y} \frac{\sqrt{y}}{y + x}$$

is a mild function of its argument, with $K(0) = 1$. The exponential factor dominates the dependence on $k_E^2 D_\parallel t$. The first term is the contribution of the pole, as mentioned. Further Fourier-transforming to real space gives

$$\begin{aligned} G(\mathbf{y}, t; 0|0) = {} & G(\mathbf{y}, t; 0|0; E = 0)[\theta(-f_1)\, k_E(\pi D_\parallel t)^{1/2} \\ & + K(k_E^2 D_\parallel t/4) \exp(-k_E^2 D_\parallel t/4)] \end{aligned} \tag{5.19}$$

where $G(\mathbf{y}, t; 0|0; E = 0)$ is just

$$\left(\frac{1}{4\pi D_\perp t}\right)^{(d-1)/2} \left(\frac{1}{4\pi D_\parallel t}\right)^{1/2} e^{-y^2/4D_\perp t}$$

$D_\parallel$ couples to k_E and $D_\perp$ couples to k^2, as expected. Similar argument used in Section 4 can now be applied for a qualitative understanding of the anomalous relaxation. It goes as follows:

1. For sufficiently large and positive f_1 (e.g., Fig. 6a), the level of nonlocality is suppressed severely by the factor $\exp(-k_E^2 D_\parallel t/4)$, leading to a truncation of the important range of nonlocality down to $O(k_E^{-1})$, both spatially and temporally. Thus, for long-wavelength deformation $k^{-1} \gg k_E^{-1}$, the kinetics occurring near the interface approaches that of model A for large E. This explains why $\omega(k_v) \sim -k_v^2$.

2. For the opposite configuration with finite and negative f_1 (an extreme example is as shown in Fig. 6b), the overall strength of nonlocality is enhanced by a factor of $(k_E D_{\|} t)^{1/2}$ for time and length scale $(D_{\|} t)^{1/2} > k_E^{-1}$ and $y > k_E^{-1}$. For a sinusoidal deformation of wavelength k^{-1}, the important coupling to the relaxation at any given point comes from a region of size k^{-1} and time $(Dk^2)^{-1}$ centered about that point. The relaxation is therefore significantly slowed down, as evidenced by the way these nonlocal couplings enter the equation of motion (5.5).

At the microscopic level, we indeed expect physically a configuration with $f_1 > 0$ to relax more rapidly than one with $f_1 < 0$, because the particles near the interface for $f_1 > 0$ have more neighboring vacancies to move into along the direction of E.

6. DECAY FROM AN INITIAL DEFORMATION

In Section 4 it was shown that the planar interface parallel to E is stable against small deformations of all wavelengths. Here we examine explicitly how a tilted interface relaxes from an initial configuration in the absence and presence of the driving force. The possible effect of the boundary is also discussed.

We consider an initially flat interface ($f = 0$ for $t < 0$). A sudden kick at $t = 0$ deforms the interface into a shape given by the function $f(\mathbf{y}, t = 0)$. We are interested in how the interface relaxes toward the planar configuration. Before we go on, let us emphasize that solving an initial value problem for an integral equation may lead to mathematical difficulties. In contrast to solving a differential equation, specifying the initial conditions, namely the function f and its derivatives at $t = 0$, does not in general lead to a complete knowledge of the function for $t > 0$. To acquire such knowledge we need in principle the complete history of f for all $t < 0$. Therefore, in the hope of gaining some qualitative understanding of the relaxational properties of the interface, it is intructive to consider even such a somewhat unrealistic sudden action on f at $t = 0$, in order to specify f for all $t < 0$.

To find $f(\mathbf{y}, t)$, we will use the method of Laplace transform. Let us first define

$$\tilde{f}(\mathbf{k}, s) = \int_0^\infty dt\, e^{-st} f(\mathbf{k}, t)$$

as the Laplace transform of f, after Fourier transforming with respect to the $(d-1)$-dimensional coordinates $\mathbf{y}$. Using the fact that the Laplace transform of $\dot{f}(\mathbf{k}, t)$ is $-f(\mathbf{k}, 0) + s\tilde{f}(\mathbf{k}, s)$, our linearized interface equation gives

$$\tilde{f}(\mathbf{k}, s) = \frac{1}{s + \xi k^2 \tilde{G}(\mathbf{k}, s; 0\,|\,0)^{-1}} f(\mathbf{k}, t = 0) \tag{6.1}$$

where

$$\tilde{G}(\mathbf{k}, s; 0\,|\,0) = \frac{1}{D} \frac{1}{(k^2 + D^{-1}s + ik_E k_{||})^{1/2} + (k^2 + D^{-1}s - ik_E k_{||})^{1/2}} \tag{6.2}$$

Note that $\tilde{f}(\mathbf{k}, s)/f(\mathbf{k}, t=0)$ is just the linear response function to an external field that would couple to f linearly in a Hamiltonian. An example of such a field is an external magnetic field for spin systems. The pole of the response function determines the relaxation mode as found in Section 3. The time development of f is given by the inverse Laplace transform

$$\begin{aligned} \frac{f(\mathbf{k}, t)}{f(\mathbf{k}, 0)} = \int_C \frac{ds}{2\pi i} e^{st} \{ s + D^{1/2} \xi k^2 [(Dk^2 + s + iDk_E k_{||})^{1/2} \\ + (Dk^2 + s - iDk_E k_{||})^{1/2}] \}^{-1} \\ \equiv I_c(\mathbf{k}, t) \end{aligned} \tag{6.3}$$

where C is a contour in the complex s plane, lying to the right of all singularities of $\tilde{f}(\mathbf{k}, s)$. Note that $I_c(\mathbf{k}, t=0) = 1$ and $I_c(\mathbf{k}, t<0) = 0$, as should be the case.

From now on, in this section, we restrict ourselves to two dimensions, so $\mathbf{x} = (y_{||}, z)$. We wish to consider the evolution of an initial shape that reduces to that of a planar tilted interface at sufficiently small length scale. Two examples that are simple enough for computing the inverse transforms are (see Fig. 7)

$$f(y_{||}, 0) = y_{||} e^{-\kappa |y_{||}|}$$

and

$$f(y_{||}, 0) = (1/\kappa) \sin \kappa y_{||}$$

On a length scale $\ll \kappa^{-1}$, both approximate a tilted interface as $f(y_{||}, 0) \approx y_{||}$.

We focus on the time evolution of the central planar portion of the interface, whose size is much smaller than κ^{-1}. Hence, we consider the region $|y_{||}| \ll \kappa^{-1}$. To include all important nonlocal interaction, we need

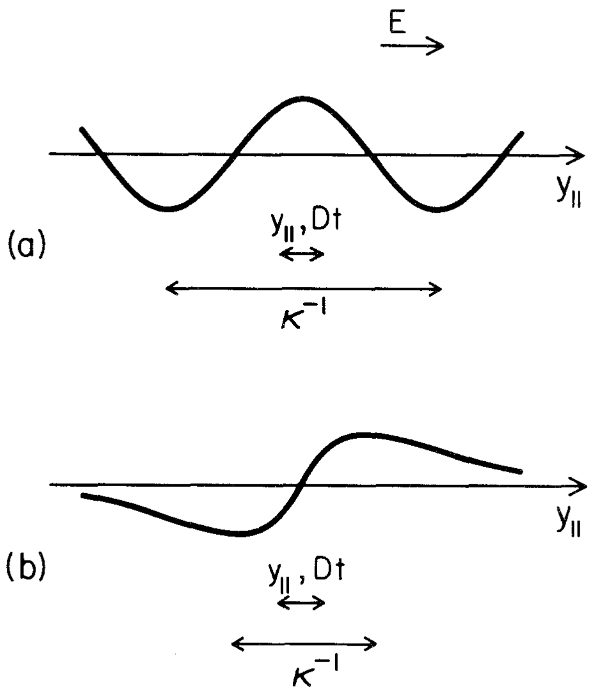

Fig. 7. The initial shapes of the deformed interface chosen for calculating their time evolution. (a) A sinusoidal shape, which is preserved under time development; and (b) a shape consistent with the periodic boundary condition. The separation of length scales is shown schematically.

to consider t such that $(Dt)^{1/2} \sim |y_{\|}|$. Thus, for large E, we have the following order of the length scales:

$$\xi \ll k_E^{-1} \ll (|y_{\|}|, (Dt)^{1/2}) \ll \kappa^{-1}$$

The important Fourier modes contributing to the time evolution of the planar portion are clearly those with $|k_{\|}| \leqslant \kappa$. With the wide separation of length scales in mind, the contour integral of (6.3) is computed as the difference of the pole contribution and those along the branch cuts. For comparison, we have also computed for $E=0$, which is listed together with that of $E \neq 0$ as follows:

$$I_c(k_{\|}, t) = I_{\text{pole}}(k_{\|}, t) + I_{\text{branch}}(k_{\|}, t)$$

$$I_{\text{pole}}(k_{\|}, t) = \begin{cases} (1 + Dk_{\|}^4 \xi^2/p + \cdots) \exp(pt) & (E \neq 0) \\ (1 - |k_{\|}| \xi + \cdots) \exp(-2k_{\|}^3 \xi Dt) & (E = 0) \end{cases} \tag{6.4}$$

where $p = -2(2k_E)^{1/2}\xi D k_{||}^{5/2} + \cdots$. The branch-cut integrals are

$$I_{\text{branch}}(k_{||}, t)$$

$$\approx \begin{cases} \dfrac{-2k_{||}^2\xi}{\pi} [\exp(-k_{||}^2 Dt)] \dfrac{1}{(k_E k_{||})^{1/2}} \displaystyle\int_0^\infty d\rho \exp(-\rho k_E |k_{||}| Dt) \dfrac{\sqrt{\rho}}{(1+\rho^2)^2} \\ \quad \times [(1-\rho^2)\cos(k_E|k_{||}|Dt) + 2\rho \sin(k_E|k_{||}|Dt)] \qquad (E \neq 0) \\ \dfrac{2}{\pi}|k_{||}|\,\xi[\exp(-k_{||}^2 Dt) \displaystyle\int_0^\infty d\rho\, [\exp(-\rho k_{||}^2 Dt) \dfrac{\sqrt{\rho}}{(1+\rho)^2} \qquad (E=0) \end{cases} \tag{6.5}$$

The results for the two initial shapes are then as follows.

1. $f(y_{||}, 0) = (1/\kappa)\sin \kappa y_{||}$: Its Fourier transform is simply

$$f(k_{||}, 0) = (\pi/i\kappa)[\delta(k_{||} - \kappa) - \delta(k_{||} + \kappa)]$$

which gives the shape-preserving evolution:

$$f(y_{||}, t) = \int_{-\infty}^{+\infty} \frac{dk_{||}}{2\pi} e^{ik_{||}y_{||}} f(k_{||}, 0)\, I_c(k_{||}, t)$$

$$= I_c(\kappa, t)\frac{1}{\kappa}\sin \kappa y_{||} \tag{6.6}$$

A natural dimensionless parameter emerges: $k_E\kappa Dt$. Thus, in the large-E limit in the sense that $k_E\kappa Dt \gg 1$ [note that this equals $k_E(Dt)^{1/2}\,\kappa(Dt)^{1/2}$, the first factor $\gg 1$, while the second $\ll 1$], we evaluate the integrals over ρ by expanding in this small parameter for $E \neq 0$, and in $\kappa^2 Dt \ll 1$ for $E = 0$, to get

$$I_c(\kappa, t) = \begin{cases} 1 - (4/\sqrt{\pi})\,\kappa\xi(\kappa^2 Dt)^{1/2} + \cdots & (E=0) \\ 1 - 2(k_E\kappa)^{1/2}\xi\kappa^2 Dt + \cdots & (E \neq 0) \end{cases} \tag{6.7}$$

Here the two terms for $E \neq 0$ comes from the pole integral, whereas for $E = 0$ the "1" comes from the pole piece and the second term from the branch-cut piece. As $k_E\kappa Dt \gg 1$, (6.7) explicitly displays the faster relaxation in the presence of E.

2. $f(y_{||}, 0) = y_{||}\exp(-\kappa\,|y_{||}|)$: Its Fourier transform is

$$f(k_{||}, 0) = -4i\kappa k_{||}/(k_{||}^2 + \kappa^2)^2 \tag{6.8}$$

which is problematic at $k_{||} = 0$ as $\kappa \to 0$; this explains the necessity of dealing with a shape converging at $y_{||} = \pm\infty$, rather than just $f \propto y_{||}$, in order to study the time evolution of the tilted interface. Here we are less fortunate than in case 1 because the parameter $k_E|k_{||}|Dt$ in the integrand of $\int d\rho$ in (6.5) is not necessarily large, as $|k_{\|}|$ is integrated from 0 to $O(\kappa)$. For this reason the computation for $k_E \kappa Dt \gg 1$ is not yet successful, so we only report that for the limit $k_E \kappa Dt \ll 1$. Note that for large k_E this corresponds to the small-time limit, namely $1 \ll k_E (Dt)^{1/2} \ll 1/\kappa (Dt)^{1/2}$. In contrast to case 1, where the influence of E on the relaxation shows up at the lowest order nontrivial terms, here we must compute the second-order effects. We focus on the planar portion as before by expanding the factor $\exp(ik_{\|} y_{\|})$ into $1 + ik_{\|} y_{\|} + \cdots$. By symmetry all even terms vanish on integrating over $k_{\|}$. The results are

$$f(y_{\|}, t) = \begin{cases} y_{\|}[1 - c_1 \xi \kappa^2 (Dt)^{1/2} - c^2 \xi \kappa^3 Dt + \cdots] + O(y_{\|}^3) & (E = 0) \\ y_{\|}[1 - c_1 \xi \kappa^2 (Dt)^{1/2} - c_2' \xi (k_E \kappa)^{1/2} \kappa^2 Dt + \cdots] + O(y_{\|}^3) & (E \neq 0) \end{cases} \tag{6.9}$$

where the coefficients c_1, c_2, and c_2' are all of order unity. Similar to case 1, faster relaxation for $E \neq 0$ is exhibited in the rotation of the interface toward the configuration parallel to E (the bracketed quantities are the tangent of the inclination).

Note that in the limit $\kappa \to 0$, corresponding to the curved portions of the interface being infinitely far away from our region of interest, we get from both case 1 and 2 the result $f(y_{\|}, t) = f(y_{\|}, 0) = y_{\|}$. This corresponds to a zero mode of the linearized equation of motion. However, although any f linear in the coordinates is a stationary solution to the full, nonlinear equation (2.10) (in fact, it is the only family of nontrivial solutions), it is unlikely to remain a zero mode, as the nonlinearity in G significantly modifies the linearized dynamics for large deviation from a planar interface, such as $f = y_{\|}$.

On the other hand, the appearance of this zero mode is fully expected for $E = 0$. We can easily show that the equation of motion is rotational invariant when $E = 0$, and that the relaxation modes $\omega(\mathbf{k})$ for deformation about a planar interface are independent of its angle, i.e., the planar interface is *equally* stable for any orientation. For $E \neq 0$, E apparently breaks the rotational invariance and its induced anisotropy is manifested in $\omega(\mathbf{k})$, as shown in Section 5.

Two comments are in order here:

1. Physically we can interpret the presence and the influence of the curved portion (represented by $y_{\|}^3$ terms) on the planar part as the boundary effects of a finite system on a tilted interface. The largest length κ^{-1} then corresponds to the system size. The limit $\kappa \to 0$ is then the situation when we only look at time and length scales very much smaller than that

of the system size, so that the effect on the time evolution of a tilted interface is not yet felt. In fact, case (2) seems to model the effect of the periodic boundary condition (PBC) along $y_{||}$. In the context of the linearized equation, this then implies that PBC has a stabilizing effect on the interface parallel to E. Thus, our result indicates that the planar interface parallel to E for $T < T_c$ found in computer simulations may merely be an effect of the periodic boundary condition imposed on the density variable ϕ. Whether this remains true including nonlinearities requires more extensive examination.

2. Another point to note is that for finite κ the faster rotation of the interface toward its parallel position appears only for $k_E > \kappa$. This is intuitively obvious if we interpret κ^{-1} as the finite size of the system. The particles would not notice the presence of E if E is so small that its associated length is greater than the system size.

7. ROUGHNESS OF THE INTERFACE

Although we have shown that the planar interface is stable against small deformation, it could be rough even at low temperature due to thermal fluctuations. The degree of the roughness is conventionally measured by the quantity

$$\omega^2 \equiv G(\mathbf{y}, t \mid \mathbf{y}, t) = \langle f^2(\mathbf{y}, t) \rangle \tag{7.1}$$

where $\langle \cdots \rangle$ denotes an average over noise in the stationary state.

The roughness of the interface depends on the number of dimensions of the system. It is important to determine the exponent that characterizes the divergence of w with the system size L: $w \sim L^{\theta}$. When $\theta < 1$, we have $w/L \to 0$ as $L \to \infty$, so that the interface is effectively smooth. The lower critical dimensionality d_l can be determined by the condition $\theta(d_l) = 1$,[18] which is when the interface wanders all over the system so that it destroys the ferromagnetic order. For example, for continuum models, w diverges for $d \leqslant 3$ in the pure Ising model.[16] At $d = 3$ it diverges logarithmically. These are true for all temperatures below T_c. When lattice effects are taken into account, one finds[17] that there exists at $d = 3$ a roughening transition temperature $0 < T_R < T_c$ such that the interface is rough only for $T_R < T < T_c$. Thus, the continuum interface model fails below T_R. However, we are concerned here neither with the effect of a lattice nor with that of the roughening transition.

In the following we calculate the width w for the driven diffusive model, starting from the linearized equation of motion for an interface parallel to E. We only managed to work at the linear order in f. The ignored nonlinearities come from two sources: the curvature on the lhs of (2.10) and that contained in G. The correction for the corresponding model A calculation due to the former is argued[16] to be merely an amplitude change. Similarly, those nonlinearities in G for model B do not change the exponent θ obtained from the linear equation, since model B shares the same statics with model A. Unfortunately, we are not yet able to produce a similar proof for the driven model. Let us proceed and see what we can get.

To calculate any correlation function, we need the correlator of the noise $\eta(\mathbf{y}, t)$ in (2.10). This can be derived in a straightforward way in the linear approximation if we know that of the bulk noise ζ. The interfacial noise is given by

$$\eta(\mathbf{y}, t) = \frac{1}{2\phi_\infty} \int d^{d-1}y'\, dz'\, dt'\, G(\mathbf{y}, f, t \mid \mathbf{y}', z', t')\, \zeta(\mathbf{y}', z', t') \tag{7.2}$$

The $\langle \eta\eta' \rangle$ is most conveniently expressed in the momentum-frequency space. For the bulk noise, we have

$$\begin{aligned}&\langle \zeta(\mathbf{k}, \omega; z)\, \zeta(\mathbf{k}', \omega'; z') \rangle \\ &\quad = -2\lambda(k_\parallel^2 + \gamma k_\perp^2 - \gamma \partial_z^2)\, \delta(z-z')(2\pi)^d \delta^{d-1}(\mathbf{k}+\mathbf{k}')\, \delta(\omega+\omega')\end{aligned} \tag{7.3}$$

where as before $\mathbf{k} = (k_\parallel, \mathbf{k}_\perp)$ is $(d-1)$-dimensional. The parameter γ accounts for anisotropy induced by E. After linearization by setting $f=0$ in G, we use

$$\eta(\mathbf{k}, \omega) = \frac{1}{2\phi_\infty} \int dz'\, G(\mathbf{k}, \omega; 0 \mid z')\, \zeta(\mathbf{k}, \omega; z') \tag{7.4}$$

from (7.2), and the equation obeyed by G,

$$[-i\omega + Dk^2 - D\partial_{z'}^2 - E\phi_c(z')\, ik_\parallel]\; G(\mathbf{k}, \omega; 0 \mid z') = \delta(z') \tag{7.5}$$

to obtain, after a little algebra,

$$\begin{aligned}&\langle \eta(\mathbf{k}, \omega)\, \eta(\mathbf{k}', \omega') \rangle \\ &\quad = \frac{\lambda\gamma}{4D\phi_\infty^2} (2\pi)^d\, \delta^{d-1}(\mathbf{k}+\mathbf{k}')\, \delta(\omega+\omega') \\ &\qquad \times \Big\{ 2Dk_\parallel^2(\gamma^{-1}-1) \int dz'\, G(\mathbf{k}, \omega; 0 \mid z')\, G(-\mathbf{k}, -\omega; 0 \mid z') \\ &\qquad + G(\mathbf{k}, \omega; 0 \mid 0) + G(-\mathbf{k}, -\omega; 0 \mid 0) \Big\}\end{aligned} \tag{7.6}$$

Using the explicit expression for G, one finds for the integral on the rhs

$$\int dz'\, G(\mathbf{k}, \omega; 0\,|\,z')\, G(-\mathbf{k}, -\omega; 0\,|\,z')$$
$$= [(k^2 - iD^{-1}\omega + ik_E k_{||})^{1/2} + (k^2 + iD^{-1}\omega - ik_E k_{||})^{1/2}]^{-1}$$
$$\times [(k^2 - iD^{-1}\omega - ik_E k_{||})^{1/2} + (k^2 + iD^{-1}\omega + ik_E k_{||})^{1/2}]^{-1} \tag{7.7}$$

Thus, in the absence of the anisotropy, $\gamma \to 1$, the correlation of the noise is just given by the Green's function, even though we started with a δ-function correlation in the bulk noise. It follows that the constraint of local conservation generates finite-range correlation in the interfacial noise.

Let us now derive the correlation function and the linear response function of f using (2.10) and (7.6). We can either get these from the equations of motion of $G_{ff}(\mathbf{y}, t) \equiv \langle f(\mathbf{y}, t)\, f(\mathbf{0}, 0)\rangle$ and that of $\langle f(\mathbf{y}, t)\rangle$ in the presence of an external field, or equivalently from the generating functional (the Martin–Siggia–Rose functional) as in Ref. 6. Whichever method one uses, the results are

$$G_{\tilde{f}f}(\mathbf{k}, \omega) = \frac{1}{-i\omega + k^2 \xi G^{-1}(\mathbf{k}, \omega)} \tag{7.8}$$

$$G_{ff}(\mathbf{k}, \omega) = \frac{\lambda\gamma}{4D\phi_\infty^2 \xi k^2}\,[1 + 2(\gamma^{-1} - 1)\, k_{||}^2 F(\mathbf{k}, \omega)]$$
$$\times [G_{\tilde{f}f}(\mathbf{k}, \omega) + G_{\tilde{f}f}(-\mathbf{k}, -\omega)] \tag{7.9}$$

where

$$G^{-1}(\mathbf{k}, \omega) = D[(k^2 - iD^{-1}\omega + ik_E k_{||})^{1/2} + (k^2 - iD^{-1}\omega - ik_E k_{||})^{1/2}] \tag{7.10}$$

and

$$F(\mathbf{k}, \omega) = [(k^2 - iD^{-1}\omega + ik_E k_{||})^{1/2} + (k^2 + iD^{-1}\omega - ik_E k_{||})^{1/2}]^{-1}$$
$$\times [(k^2 - iD^{-1}\omega - ik_E k_{||})^{1/2} + (k^2 + iD^{-1}\omega + ik_E k_{||})^{1/2}]^{-1} \tag{7.11}$$

To establish a relation between $G_{\tilde{f}f}$ and the linear response function, we note that an external field $h(\mathbf{y}, t)$ that would couple to f in a Hamiltonian couples to $\tilde{f}$ in an MSR action. From this, for the linear response defined by

$$\langle f(\mathbf{y}, t)\rangle = \int_{-\infty}^{+\infty} dt' \int d^{d-1}y'\, R(\mathbf{y}, t\,|\,\mathbf{y}'t')\, h(\mathbf{y}', t') + O(h^2) \tag{7.12}$$

it follows that

$$R(\mathbf{k}, \omega) = (\xi/\sigma)\, G^{-1}(\mathbf{k}, \omega)\, G_{\tilde{f}f}(\mathbf{k}, \omega) \tag{7.13}$$

where σ is the macroscopic surface tension. The fluctuation-dissipation theorem $G_{ff}(\mathbf{k}, \omega) \propto (1/\omega) \operatorname{Im} R(\mathbf{k}, \omega)$ is satisfied only if the anisotropy is removed: $\gamma = 1$. This agrees with the finding in Ref. 6 regarding the critical behavior.

Returning to our question of roughness, to evaluate $\int d\omega$, we note from Section 6 that

$$\frac{f(\mathbf{k}, t)}{f(\mathbf{k}, 0)} = \int \frac{d\omega}{2\pi} e^{-i\omega t}[G_{\tilde{f}f}(\mathbf{k}, \omega) + G_{\tilde{f}f}(-\mathbf{k}, -\omega)]$$

Hence it is unity for $t = 0$. This immediately gives

$$G_{ff}(\mathbf{k},\ t=0) = \frac{\lambda\gamma}{4D\phi_\infty^2 \xi k^2} [1 + 2(\gamma^{-1} - 1)\, k_{\|}^2 I(\mathbf{k})] \tag{7.14}$$

where

$$I(\mathbf{k}) = \int \frac{d\omega}{2\pi} F(\mathbf{k}, \omega)[G_{\tilde{f}f}(\mathbf{k}, \omega) + G_{\tilde{f}f}(-\mathbf{k}, -\omega)] \tag{7.15}$$

Though $I(\mathbf{k})$ is complicated to evaluate, one realizes $I(\mathbf{k} \to 0) = 0$, so the infrared property of the first term in the square brackets of (7.14) dominates. Thus,

$$\begin{aligned} \omega^2 &= G_{ff}(\mathbf{y} = \mathbf{0}, t = 0) \\ &= \int_{L^{-1}} \frac{d^{d-1}k}{(2\pi)^{d-1}} G_{ff}(\mathbf{k}, t = 0) \\ &\sim \int_{L^{-1}} \frac{d^{d-1}k}{k^2} \sim L^{3-d} \end{aligned} \tag{7.16}$$

giving $d_l = 1$ as for $E = 0$. In computing $\int d^{d-1}k$, we have *assumed* that the dominant fluctuation has $k_{\|} \sim k_{\perp}$, which is unlike that in the ciritical theory, where $k_{\|} \sim k_{\perp}^2$. This assumption is consistent with the range of non-locality contained in G, wich gives $\exp[-(y_{\|}^2 + y_{\perp}^2)/4Dt]$ apart from a multiplicative enhancement factor $(1/k_E |y_{\|}|) \exp(k_E |y_{\|}|/2)$.

The question of whether the interface is effectively smooth in two dimensions, in the sense that $\omega/L \to 0$ as $L \to \infty$, is clearly important in interpreting numerical results. For a system size of order 30, the roughness of the interface is negligibly small at the resolution of 1 (i.e., a lattice site). So it is difficult to draw any conclusion regarding the roughness of the interface from available numerical findings at two dimensions. More extensive numerical work focused on the interface is needed to test the validity of this linear result.

8. CONCLUSION

Here we briefly summarize our results: a central feature of the interaction of the interfacial degrees of freedom is that of nonlocality—the motion of any given part on the interface couples to that of all others, with the strength of coupling decaying exponentially to zero at long distance. For small deformations of the interface, we found no instability to drive the system toward the nonlinear regime. Thus, for our purposes we are justified in working with the linearized version of the highly nonlinear equation.

Starting from a kinetic equation of the tilted planar interface, we were able to show that the interface is stable for small deformations of all wavelengths. It is always stable (marginally) against normal translation, which represents the zero mode of the translational invariance of the bulk equation. We found that the relaxation of the interface has strong orientational dependence (i.e., $f_1 < 0$ is very different from $f_1 > 0$). These were understood qualitatively as a consequence of the influence of the driving force on the coupling of neighboring parts of the interface.

We demonstrated explicitly the effect of the external field on the relaxation after an initial deformation on a planar interface parallel to E. The field is seen to speed up the relaxation if it is sufficiently strong—determined by a comparison of length scales. In the case of an initial deformation consistent with the periodic boundary conditions that are used in all computer simulations, the results seems to suggest that these boundary conditions act to stabilize the interface, favoring a parallel orientation with respect to E.

The roughness of the interface parallel to E, which arises from thermal fluctuations, was shown to linear order in the interfacial position to diverge with the system size in the same way as for $E = 0$. However, the validity of this result beyond linear order is not yet proved. Should this be true generally, it would suggest that the lower critical dimensionality d_l remains as 1.

APPENDIX. DERIVATION OF THE GREEN'S FUNCTION FOR $E \neq 0$

Here we give the derivation of the Green's function for $E \neq 0$ of an interface parallel to E. The results are used in Section 3. The Green's function G is defined in Section 2 by the following equation:

$$[-\partial/\partial t - D\nabla^2 + \lambda E\phi_c(z)\partial_{\|}]\, G(p' \mid p) = \delta(p' - p) \tag{A.1}$$

where as before $p \equiv (\mathbf{r}; t) = (y_{\|}, \mathbf{y}_{\perp}, z; t)$, with $\mathbf{y}_{\perp}$ being $(d-2)$-dimensional. For arbitrary profile $\phi_c(z)$, it would be difficult to solve for G. To make progress, we make a low-temperature approximation

$$\phi_c(z) \approx \phi_\infty[2\theta(z - z_0) - 1] \tag{A.2}$$

of the same nature as the approximation $\phi_c^2(z) \approx \phi_\infty^2$ made in deriving the linearized bulk equation in Section 2. In doing this, we are limiting ourselves to situations where the interesting phenomena occur at length scales $\gg \xi$, the width of the interface. However, since interfacial properties are generally not sensitive to the details of the density profile, (A.2) should still capture the essential physics that arises from the existence of an interface separating two opposite phases.

Since the planar interface can be translated freely along its normal direction, we can arbitrarily define its position as $z = 0$, so $z_0 = 0$. To simplify, we first Fourier-transform all coordinates but z to get from (A.1)

$$[-i\omega + Dk^2 - D\partial_z^2 - \lambda E\phi_c(z)\, ik_{\|}]\, G(\mathbf{k}, \omega; z' \,|\, z) = \delta(z' - z) \tag{A.3}$$

where $\mathbf{k}$ is $(d-1)$-dimensional. We now solve G subject to the boundary condition that $G \to 0$ as $|z|$ or $|z'|$ goes to infinity. We expect that G will be a linear combination of exponentials. We have to consider separate regions in z and z' and match the solutions at boundaries. First suppose $z' < 0$. Let

$$V = -i\omega + Dk^2, \qquad v = -\lambda E\phi_\infty ik_{\|} \tag{A.4}$$

Using (A.2), we thus have the following:

1. $z > 0$: G obeys

$$(-D\partial_z^2 + V + v)G = 0$$

Substituting $G = Ae^{-pz}$, we get $p^2 = D^{-1}(V + v)$ with $\operatorname{Re} p > 0$.

2. $z < 0$: G obeys

$$(-D\partial_z^2 + V - v)G = \delta(z' - z)$$

Guided by symmetry with respect to $z \leftrightarrow z'$, we write

$$G = B\exp(-q\,|z - z'|) + C\exp[q(z + z')]$$

a direct substitution of which into the differential equation shows that $q^2 = D^{-1}(V - v)$ with $\operatorname{Re} q > 0$. This expression already satisfies the continuity of G at $z = z'$. The discontinuity of the first derivative of G is obtained by integrating z from $z' - \varepsilon$ to $z' + \varepsilon$ with infinitesimal ε, which gives

$$D[G'(z = z' - \varepsilon) - G'(z = z' + \varepsilon)] = 1$$

so $B = 1/(2Dq)$. The coefficients A and C are determined by matching the boundary conditions at $z = 0$:

$$Be^{qz'} + Ce^{qz'} = A$$
$$-qBe^{qz'} + qCe^{qz'} = -pA$$

which give

$$C = \frac{q-p}{q+p} B$$
$$A = (B + C)\, e^{qz'} = \frac{2q}{q+p} Be^{qz'}$$

Putting all these together, we obtain for $z' < 0$

$$G(\mathbf{k}, \omega; z' \mid z) = \begin{cases} \dfrac{1}{D(q+p)} e^{-pz+qz'} & z > 0 \\ \dfrac{1}{2Dq}\left(e^{-q|z-z'|} + \dfrac{q-p}{q+p} e^{q(z+z')}\right) & z < 0 \end{cases} \tag{A.5}$$

where we recall that both p and q have positive real parts for real ω:

$$\begin{aligned} p &= (k^2 - D^{-1} i\omega + ik_E k_{||})^{1/2} \\ q &= (k^2 - D^{-1} i\omega + ik_E k_{||})^{1/2} \end{aligned} \tag{A.6}$$

with $k_E \equiv \lambda E D^{-1} \phi_\infty$ and their branches are defined as in Section 3.

The results for $z' > 0$ are obtained from (A.5) by replacing p by q and vice versa, and making the changes $z \to -z$ and $z' \to -z'$.

In a linear analysis we only need the following quantity:

$$G(\mathbf{k}, \omega; 0 \mid 0) = 1/D(q+p) \tag{A.7}$$

ACKNOWLEDGMENTS

The author is grateful to Prof. John Cardy for his constant guidance and numerous stimulating discussions. He would also like to thank Prof. James Langer and Prof. Royce Zia for their interests and useful comments. This work was supported by the National Science Foundation under grant PHY83-13324.

REFERENCES

1. S. Katz, J. L. Lebowitz, and H. Spohn, *Phys. Rev. B* **28**:1655 (1983); **34**:497 (1984); *J. Stat. Phys.* **38**:725 (1985).
2. H. van Beijeren and L. Schulman, *Phys. Rev. Lett.* **53**:806 (1984).
3. H. van Beijeren, R. Kutner, and H. Spohn, *Phys. Rev. Lett.* **54**:2026 (1985).
4. H. K. Janssen and B. Schmittmann, *Z. Phys. B* **63**:517 (1986).
5. K. Gawadzki and A. Kupiainen, *Nucl. Phys. B* **269**:45 (1986).
6. K.-t. Leung and J. L. Cardy, *J. Stat. Phys.* **44**:567 (1986).
7. H. K. Janssen and B. Schmittmann, *Z. Phys. B* **64**:503 (1986).
8. J. V. Vallés and J. Marro, *J. Stat. Phys.* **43**:411 (1986).
9. J. Krug, J. L. Lebowitz, H. Spohn, and M. Zhang, *J. Stat. Phys.* **44**:535 (1986).
10. J. Honkonen and A. Kupiainen, preprint.
11. J. B. Bates, J.-C. Wang, and N. J. Dudney, *Phys. Today* **1982**(July):46.
12. J. B. Bates and G. C. Farrington, eds., *Fast Ionic Transport in Solids* (North-Holland, New York); *Solid State Ionics* **5** (1981); W. Dieterich, P. Fulde, and I. Peschel, *Adv. Phys.* **29**:527 (1980).
13. J. S. Langer and L. A. Turski, *Acta Metall.* **25**:1113 (1977).
14. K. Kawasaki and T. Ohta, *Prog. Theor. Phys.* **67**:147 (1982).
15. J. S. Langer, *Acta Metall.* **25**:1121 (1977).
16. D. J. Wallace, in *Proc. École d'Été de Physique théorique, Les Houches, 1982* (North-Holland, Amsterdam), and references therein.
17. J. D. Weeks, G. H. Gilmer, and H. J. Leamy, *Phys. Rev. Lett.* **31**:549 (1973); S. T. Chui and J. D. Weeks, *Phys. Rev. B* **14**:4978 (1976).
18. G. Grinstein and S. Ma, *Phys. Rev. Lett.* **49**:685 (1982); *Phys. Rev. B* **28**:2588 (1983).
19. P. C. Hohenberg and B. I. Halperin, *Rev. Mod. Phys.* **49**:435 (1977).